Graphing Sine and Cosine Functions

Any equation of the form $y = a\sin\left[b\left(x-c\right)\right]+d$ with $a \neq 0$ and $b \neq 0$ is a **sine function**. Its graph is called a **sine wave**, **sinusoidal wave**, or **sinusoid.** The graph of any sine function is a transformation of the graph of $y = \sin x$.

We assume x is in radians unless the problem specifically states that it is in degrees.

As the terminal side of an angle rotates around the unit circle, how does the value of the sine change?

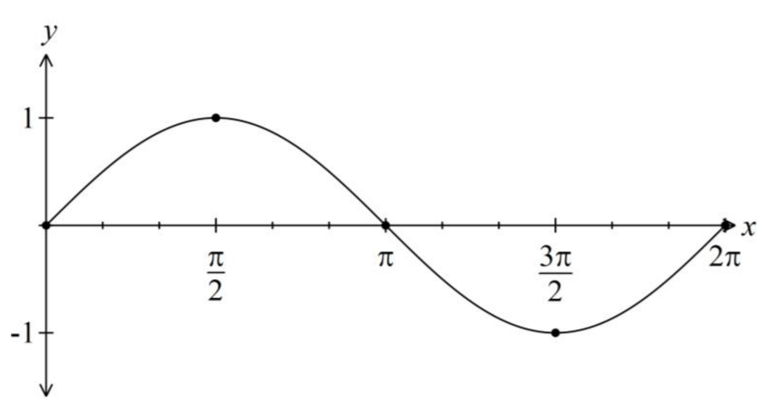
- From 0 to $\pi/2$, the sine increases from 0 to 1.
- From $\pi/2$ to π, the sine decreases from 1 to 0.
- From π to $3\pi/2$, the sine decreases from 0 to –1.
- From $3\pi/2$ to 2π, the sine increases from –1 to 0.
- The cycle repeats.

Because $\sin\left(x+2\pi\right) = \sin x$, the shape we see in the interval $\left[0, 2\pi\right]$ repeats on the intervals $\left[2\pi, 4\pi\right]$, $\left[4\pi, 6\pi\right]$, $\left[-2\pi, 0\right]$, $\left[-4\pi, -2\pi\right]$, etc.

A repeating function like $y = \sin x$ is called a **periodic function.** The length of the smallest non-repeating unit is the **period** of the function. The period of $y = \sin x$ is 2π. The graph of $y = \sin x$ over any interval of length 2π is called a **cycle.** The graph of $y = \sin x$ over $\left[0, 2\pi\right]$ is the **fundamental cycle.**

Key points on the graph of $y = \sin x$:

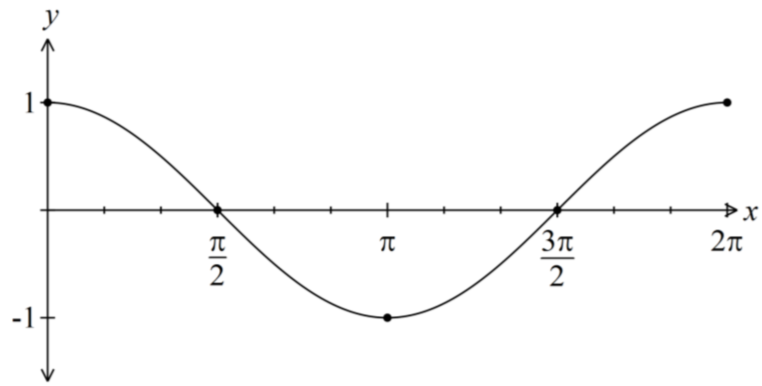
x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \sin x$	0	1	0	-1	0



The graph of $y = \cos x$ has the same shape as the graph of $y = \sin x$, but it is shifted to the left by a distance of $\pi/2$. For this reason, the graph of $y = \cos x$ is also called a sine wave. The graph of $y = \cos x$ over $[0, 2\pi]$ is called the **fundamental cycle** of $y = \cos x$.

Key points on the graph of $y = \cos x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \cos x$	1	0	-1	0	1



Domain and Range of the Cosine and Sine Functions

The two functions $f(x) = \cos x$ and $g(x) = \sin x$, both trigonometric functions of *x*, are defined for all real values of *x*. This is evident in the preceding graphs of cosine and sine. Or, going back to our Unit Circle definitions for cosine and sine, whether we think of identifying the real number *t* with the angle radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both cosine and sine are defined for any real input number *t*. Thus, the **domain** of both $f(t) = \cos t$ and $g(t) = \sin t$ is $(-\infty, \infty)$**.**

Looking at the graphs of the cosine and sine functions, we see that the range includes all real numbers between –1 and 1, inclusive. Revisiting the Unit Circle, $f(t) = \cos t$ and $g(t) = \sin t$ represent x and y-coordinates, respectively, of points on the Unit Circle. As points on the Unit Circle, they take on all of the values between –1 and 1, inclusive. In other words, the **range** of both $f(t) = \cos t$ and $g(t) = \sin t$ is the interval $[-1,1]$. Following is a summary of properties of both functions.

Theorem 3.1. Properties of the Cosine and Sine Functions:

The function $f(t) = \cos(t)$

- has domain $(-\infty, \infty)$
- has range $[-1, 1]$
- is continuous and smooth
- is even
- has period 2π

The function $g(t) = \sin(t)$

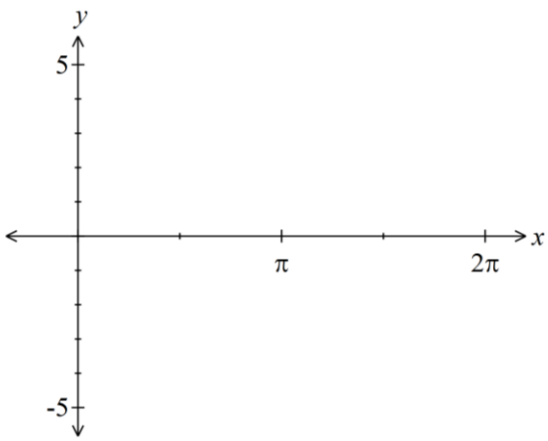
- has domain $(-\infty, \infty)$
- has range $[-1, 1]$
- is continuous and smooth
- is odd
- has period 2π

The effect of changing the value of *a*:

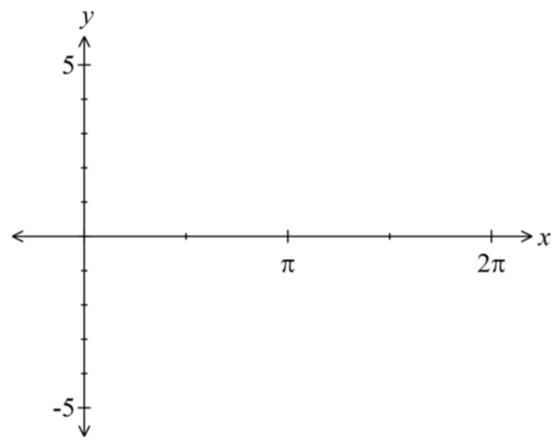
The **amplitude** of $y = a \sin x$ or $y = a \cos x$ is $|a|$. The amplitude is the “height” of the sine wave. It is half the difference between the maximum and minimum points on the graph. If a is negative, the graph is reflected over the *x*-axis.

Examples: Sketch the graphs of the following and determine the amplitude and range of each.

$y = 3\sin x$



$y = -5\cos x$

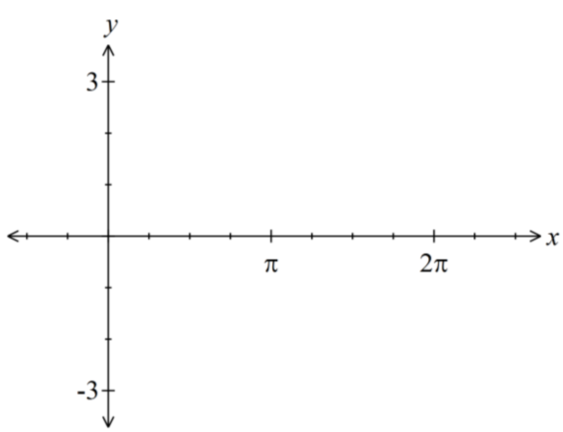


The effect of changing the value of *c*:

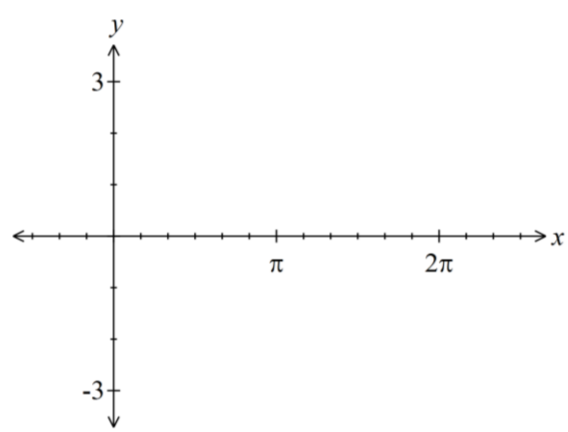
The **phase shift** of the graph of $y = \sin(x - c)$ or $y = \cos(x - c)$ is *c*. Notice that the sign of *c* is the opposite of the sign in the equation. This means that the graph is shifted *c* units to the right if *c* is positive, or *c* units to the left if *c* is negative.

Examples: Sketch each graph and find the amplitude, phase shift, and range of each function.

$y = \cos(x - \pi/4)$



$y = 2\sin(x + \pi/3)$

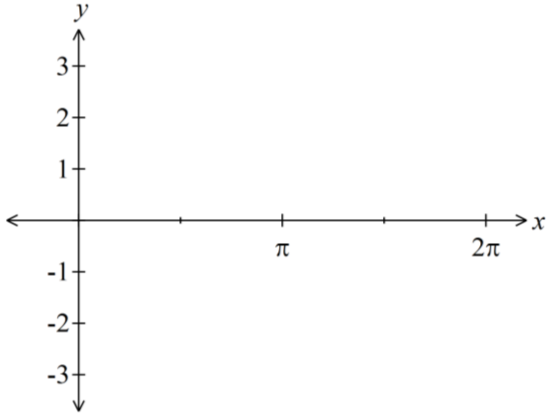


The effect of changing the value of *d*:

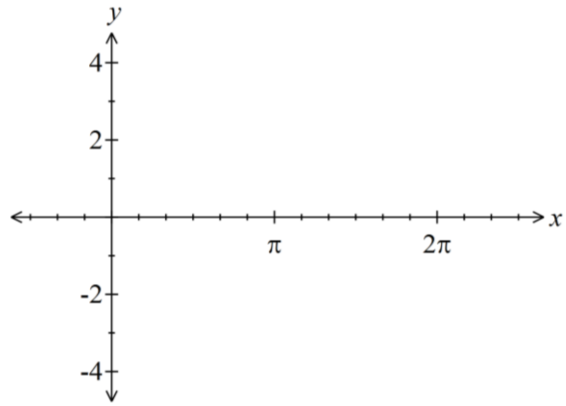
The **vertical translation** of the graph of $y = \sin x + d$ or $y = \cos x + d$ is d. This means that the graph is shifted *d* units up if *d* is positive, or *d* units down if *d* is negative.

Examples: Sketch each graph and find the amplitude, phase shift, vertical shift, and range of each function

$y = \sin x + 2$



$y = 3\cos\left(x - \pi/6\right) - 1$

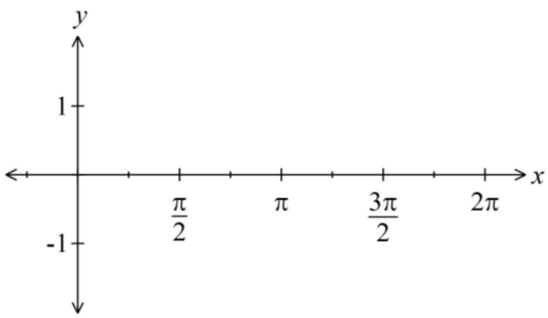


The effect of changing the value of *b*:

The **period** of the graph of $y = \sin(bx)$ or $y = \cos(bx)$ for $b > 0$ is $P = 2\pi/b$. This means that there are *b* cycles every 2π units. The **frequency**, *F*, of a sine wave with period *P* is defined by $F = 1/P = b/2\pi$.

Examples: Sketch the graphs of the following and determine the period and frequency of each.

$y = \sin\left(2x + \dfrac{\pi}{3}\right)$



$y = \cos\left(x/2\right)$